

# The Schur-Horn theorem in von Neumann algebras

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Dedicated to the memory of William Arveson(1934-2011)

## Abstract

A few years ago, Richard Kadison thoroughly analysed the diagonals of projection operators on Hilbert spaces and asked the following question: Let  $\mathcal{A}$  be a masa in a type  $II_1$  factor  $\mathcal{M}$  and let  $A \in \mathcal{A}$  be a positive contraction. Letting  $E$  be the canonical normal conditional expectation from  $\mathcal{M}$  to  $\mathcal{A}$ , can one find a projection  $P \in \mathcal{M}$  so that

$$E(P) = A?$$

In a later paper, Kadison and Arveson, as an extension, conjectured a Schur-Horn theorem in type  $II_1$  factors. In this paper, I give a proof of this conjecture of Arveson and Kadison. I also prove versions of the Schur-Horn theorem for type  $II_\infty$  and type  $III$  factors as well as finite von Neumann algebras.

## 1 Introduction

The classical Schur Horn theorem[18], [11], relates the diagonal and the eigenvalue list of a square matrix: Let  $A$  be a positive semidefinite element of  $M_n(\mathbb{C})$  and let  $d = (d_1, d_2, \dots, d_n)$  and  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  be the lists of diagonal entries and eigenvalues respectively, both sorted in non-increasing order. Then, the Schur-Horn theorem says that we must have

$$d_1 + \dots + d_k \leq \lambda_1 + \dots + \lambda_k \quad 1 \leq k \leq n \quad \text{and} \quad d_1 + \dots + d_n = \lambda_1 + \dots + \lambda_n.$$

The last fact follows from two ways of calculating the trace.

The above condition on the lists is denoted by saying that the diagonal list is majorized by the eigenvalue list, written  $d \prec \lambda$ . The Schur-Horn theorem states that further, given two positive lists with the second majorizing the first, there is a positive semi-definite matrix  $A$  with the second list as its eigenvalue list and the first as its diagonal list.

Majorization can also be defined for matrices. Given two positive operators  $A, S$  in  $M_n(\mathbb{C})$ , we say that  $A \prec S$  if the eigenvalue sequence of  $A$  is majorized by the eigenvalue sequence of  $S$ . The Schur-Horn theorem can then be stated as saying that if  $A$  is a diagonal positive matrix and  $S$  a positive matrix so that  $A \prec S$ , then there is a unitary operator  $U$  so that the diagonal of  $USU^*$  is  $A$ .

Majorization for matrices has the following alternate description due to Hardy, Littlewood and Polya[9],

**Definition 1.1** (Majorization). *Given two self-adjoint operators  $A, S$  in  $M_n(\mathbb{C})$ ,  $A$  is majorized by  $S$  iff*

$$\text{Tr}(f(A)) \leq \text{Tr}(f(S))$$

*for every continuous convex real valued function  $f$  defined on a closed interval  $[c, d]$  containing the spectra of both  $A$  and  $S$ .*

Majorization in type  $II_1$  factors[10] is described analogously, with the trace on  $M_n(\mathbb{C})$  in the definition replaced by the canonical trace  $\tau$ .

Let  $\mathcal{A}$  be a masa in a type  $II_1$  factor  $\mathcal{M}$ ; There is a unique trace preserving normal conditional expectation  $E : \mathcal{M} \rightarrow \mathcal{A}$  that is in many ways analogous to the restriction mapping onto the diagonal for elements of  $M_n(\mathbb{C})$ . Arveson and Kadison[4] showed that if  $S$  is a positive operator in  $\mathcal{M}$ , then  $E(S) \prec S$ . This fact can also be deduced from Hiai's work on stochastic maps on von Neumann algebras[10].

There are two natural generalizations of the Schur-Horn theorem to type  $II_1$  factors. The first originates in the standard interpretation of the Schur-Horn theorem as characterizing the set of all possible diagonals of a positive matrix. Let  $\mathcal{U}(\mathcal{M})$  be the set of unitary operators in  $\mathcal{M}$  and given an operator  $S$ , let  $\mathcal{O}(S)$  be the norm closure of the unitary orbit of  $S$ , i.e

$$\mathcal{O}(S) = \overline{\{USU^* \mid U \in \mathcal{U}(\mathcal{M})\}}^{\|\cdot\|}.$$

Two positive operators  $A$  and  $S$  in a type  $II_1$  factor  $\mathcal{M}$  are said to be *equimeasurable*, denoted  $A \approx S$  if  $\tau(A^n) = \tau(S^n)$  for  $n = 0, 1, 2, \dots$ . It is routine to see that the following are equivalent.

1.  $A \approx S$ .
2.  $A \in \mathcal{O}(S)$ .

The following is the first main theorem in this paper.

**Theorem 4.2** (The Schur-Horn theorem in type  $II_1$  factors I). *Let  $\mathcal{M}$  be a type  $II_1$  factor and let  $A, S \in \mathcal{M}$  be positive operators with  $A \prec S$ . Then, there is some masa  $\mathcal{A}$  in  $\mathcal{M}$  such that*

$$E_{\mathcal{A}}(S) \approx A.$$

The second generalization was conjectured by Arveson and Kadison in [4]. The second main theorem in this paper is the proof of their conjecture,

**Theorem 5.6** (The Schur-Horn theorem in type  $II_1$  factors II). *Let  $\mathcal{A}$  be a masa in a type  $II_1$  factor  $\mathcal{M}$ . If  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  are positive operators with  $A \prec S$ . Then, there is an element  $T \in \mathcal{O}(S)$  such that*

$$E(T) = A$$

One cannot escape having to take the norm closure of the unitary orbit of  $S$ , see the paper *loc.cit.* for a discussion on the necessity. In infinite dimensions, unitary equivalence cannot be determined from spectral data alone.

A special case of this, namely, that given any positive contraction  $A$  in  $\mathcal{A}$ , there is a projection  $P$  in  $\mathcal{M}$  so that  $E(P) = A$ , had been conjectured earlier by Kadison in [13], see also [14], who referred to it as the "carpenter" problem in type  $II_1$  factors.

**Remark 1.2.** *Neither of the two theorems directly implies the other. It is however, easy to see that theorem(5.6) implies theorem(4.2) when  $S$  has finite spectrum and that theorem(4.2) implies theorem(5.6) when  $A$  has finite spectrum.*

There has been methodical progress towards the resolution of Arveson and Kadison's conjecture (5.6): Argerami and Massey[2] showed that  $\overline{E(\mathcal{O}(S))}^{\text{SOT}} = \{A \in \mathcal{A} \mid A \prec S\}$ . This was improved by Bhat and Ravichandran[5], who showed that it is enough to take the norm closure. They also showed that the conjecture holds when both the operators  $A$  and  $S$  have finite spectrum. Dykema, Hadwin, Fang and Smith[6] gave a natural way to approach the problem and reduced the conjecture to a question involving kernels of conditional expectations. Using this approach, they were able to show that the conjecture holds, among other cases, for the radial and generator masas in the free group factors. However, it is unclear if their strategy can be used to settle the conjecture in full.

It must be pointed out that approximate Schur-Horn theorems are easier to obtain than exact ones. Further, it is possible that one might lose much fine structure: For instance, Kadison characterised the diagonals of projections in  $\mathcal{B}(\mathcal{H})$  and discovered an index obstruction to a sequence arising as the diagonal of a projection. This subtlety is lost when one passes to the norm closure of the set of diagonals, see [3] for a discussion.

This paper has six sections apart from the introduction. In section (2), I collect some standard facts in noncommutative measure theory. Section (3) exploits the useful observation that once we can solve the problem "locally", theorem (4.2) follows using transfinite induction. Section (4) contains the proof of theorem (4.2). Section (5) then builds upon this result to prove theorem (5.6). We then use the Schur-Horn theorem for type  $II_1$  factors to deduce theorems for type  $II_\infty$  factors in Section (6). Here I show that compared to the situation in  $\mathcal{B}(\mathcal{H})$ , thoroughly analysed by Kaftal and Weiss in [15], one is able to get easily stateable theorems. In the last section, namely Section (7) for sake of completeness, I explain the situation both in the case of general finite von Neumann algebras and type  $III$  factors.

**Acknowledgement 1.3.** *I would like to thank Junsheng Fang for telling me about this problem and for several useful discussions.*

## 2 Notation and basic relationships

There is a more concrete description of majorization in type  $II_1$  factors that is more convenient to work with, that we now describe. We will use the following nonstandard definition repeatedly: Given two subsets  $X$  and  $Y$  of  $\mathbb{R}$ , say that  $X \geq Y$  if  $X$  is to the right of  $Y$ , i.e.  $\inf_{x \in X} \geq \sup_{x \in Y}$ . We similarly define the relation  $X > Y$ . Also, given a self-adjoint operator  $S$ , we will use  $\alpha(S)$  to denote  $\inf\{x \in \sigma(S)\}$ .

Let  $A$  be a positive operator in type  $II_1$  factor. By the spectral theorem, there is a Borel measure with compact support,  $\mu$  on  $\mathbb{R}$  so that

$$\tau(A^n) = \int_{\mathbb{R}} x^n d\mu \quad n = 0, 1, \dots$$

Define the real valued function  $f_A$  on  $[0, 1]$  by

$$f_A(x) = \inf\{t \mid \tau(E_A((t, \infty))) \leq x\}.$$

This function  $f_A$  is non-increasing and right continuous. We have the identity

$$\tau(A^n) = \int_0^1 f_A(x)^n dm \quad n = 0, 1, \dots$$

where  $m$  denotes Lebesgue measure. The values of this function were denoted the generalised  $s$  numbers of  $A$  by Fack and Kosaki[8]. We however, choose to call the function  $f_A$  the spectral scale of  $A$ .

It is a standard fact[7] that one can find a projection valued measure which we denote by  $\mu_A$  on  $[0, 1]$  so that  $\tau(\mu_A(X)) = \mu(X)$  for any Borel measurable set  $X \subset [0, 1]$  and so that

$$A = \int f_A(t) d\mu_A(t) \tag{1}$$

This projection valued measure is not unique when there are atoms in the spectrum of  $A$ . However, given a positive operator  $A$ , we will fix a measure once and for all and use  $\mu_A$  to denote this.

Given two positive operators  $A$  and  $S$  inside a type  $II_1$  factor  $M$  with spectral scales  $f_A$  and  $f_S$  respectively, it can be shown that  $S$  majorizes  $A$ , written  $A \prec S$  if

$$\int_0^r f_A(x) dm \leq \int_0^r f_S(x) dm, \quad 0 \leq r \leq 1 \quad \text{and} \quad \int_0^1 f_A(x) dm = \int_0^1 f_S(x) dm \tag{2}$$

When we do not have the last trace equality, we say that  $S$  submajorizes  $A$  and denote this by  $A \prec_w S$ .

There are two concise ways of representing these inequalities in type  $II_1$  factors. The first uses the KyFan norm functions are defined by

$$F_A(x) := \int_0^x f_A(t) dm(t) \quad \text{for } 0 \leq x \leq 1 \quad (3)$$

Note that  $F_A$  is continuous. We have that,  $A \prec_w S$  iff  $F_A(x) \leq F_S(x)$  for  $x \in [0, 1]$ . If we also have that  $F_A(1) = F_S(1)$ , then,  $A \prec S$ .

Alternately,  $A \prec S$  if

$$\tau(A\mu_A([0, t]) \leq \tau(S\mu_S([0, t]) \text{ for } 0 < t < 1 \quad \text{and} \quad \tau(A) = \tau(S) \quad (4)$$

Given two positive operators  $A$  and  $S$  in a type  $II_1$  factor  $\mathcal{M}$ , define the quantity  $L_{\mathcal{M}}(A, S)$ , also denoted simply by  $L(A, S)$  by

$$L(A, S) := \min_{0 \leq t \leq 1} (F_S(t) - F_A(t)). \quad (5)$$

We have that  $A \prec_w S$  exactly when  $L(A, S) = 0$  and the function  $L(A, S)$  measures how far  $S$  is from submajorizing  $A$ . We record some facts about the quantity  $L(A, S)$ .

**Lemma 2.1.** *Let  $A, S$  be positive operators in a type  $II_1$  factor  $\mathcal{M}$ . Then,*

1.  $L(A, S) \geq -\tau(A)$ .
2. *Suppose  $A$  and  $S$  commute with a set of commuting projections  $\{P_1, P_2, \dots, P_k\}$  which sum up to  $I$ . Then,*

$$L(A, S) \geq \sum_{m=1}^k \tau(P_m) L_{P_m \mathcal{M} P_m}(AP_m, SP_m)$$

3. *Suppose additionally that*

$$\sigma_{P_1 \mathcal{M} P_1}(AP_1) \geq \sigma_{P_2 \mathcal{M} P_2}(AP_2) \geq \dots \geq \sigma_{P_k \mathcal{M} P_k}(AP_k)$$

and

$$\sigma_{P_1 \mathcal{M} P_1}(SP_1) \geq \sigma_{P_2 \mathcal{M} P_2}(SP_2) \geq \dots \geq \sigma_{P_k \mathcal{M} P_k}(SP_k)$$

Then, letting  $Q_0 = 0$  and  $Q_m = P_1 + \dots + P_m$  for  $m = 1, \dots, k$ ,

$$L(A, S) = \min_{1 \leq m \leq k} \tau((S - A)Q_{m-1}) + \tau(P_m) L_{P_m \mathcal{M} P_m}(AP_m, SP_m)$$

*Proof.* For the first assertion, we have

$$L(A, S) := \min_{0 \leq t \leq 1} (F_S(t) - F_A(t)) \geq \min_{0 \leq t \leq 1} F_S(t) - \max_{0 \leq t \leq 1} F_A(t) \geq -\tau(A)$$

For the second, it is easy to see that once we have proved the assertion for  $k = 2$ , the general case follows by induction. Assume then, that  $k = 2$ . Let  $0 < t < 1$  be arbitrary; We may write  $\mu_S([0, t]) = \mu_{SP_1}([0, a]) \oplus \mu_{SP_2}([0, b])$  and  $\mu_A([0, t]) = \mu_{AP_1}([0, c]) \oplus \mu_{AP_2}([0, d])$ . Suppose that  $a > c$  - The complementary case is handled similarly. This assumption implies that  $b < d$ . Also,  $f_{SP_1} \mid (c, a) > f_{SP_2} \mid (b, d)$  and we have that  $\tau(P_1)(a - c) = \tau(P_2)(d - b)$ . Thus,

$$\tau(P_1)\tau(SP_1\mu_{SP_1}([a, c])) - \tau(P_2)\tau(SP_2\mu_{AP_2}([b, d])) > 0$$

A simple calculation shows that

$$\begin{aligned}
F_S(t) - F_A(t) &= \tau[S(\mu_{SP_1}([0, a]) \oplus \mu_{SP_2}([0, b]))] - \tau[A(\mu_{AP_1}([0, c]) \oplus \mu_{AP_2}([0, d]))] \\
&= \tau(P_1)\tau[SP_1\mu_{SP_1}([0, a]) - AP_1\mu_{AP_1}([0, c])] \\
&+ \tau(P_2)\tau[SP_2\mu_{SP_2}([0, b]) - AP_2\mu_{AP_2}([0, d])] \\
&= \tau(P_1)\tau[SP_1\mu_{SP_1}([0, a]) - AP_1\mu_{AP_1}([0, a])] \\
&+ \tau(P_2)\tau[SP_2\mu_{SP_2}([0, d]) - AP_2\mu_{AP_2}([0, d])] \\
&+ \tau(P_1)\tau(SP_1\mu_{SP_1}([a, c])) - \tau(P_2)\tau(SP_2\mu_{AP_2}([b, d])) \\
&\geq \tau(P_1)L_{P_1\mathcal{M}P_1}(AP_1, SP_1) + \tau(P_2)L_{P_2\mathcal{M}P_2}(AP_2, SP_2)
\end{aligned}$$

We conclude that

$$L(A, S) \geq \sum_{m=1}^2 \tau(P_m)L_{P_m\mathcal{M}P_m}(AP_m, SP_m)$$

For the last assertion, given the hypotheses, it is easy to see that

$$f_A(t) = f_{AP_m} \left( \frac{t - \tau(Q_{m-1})}{\tau(P_m)} \right) \quad \text{if } \tau(Q_{m-1}) \leq t < \tau(Q_m)$$

and thus,

$$F_A(t) = \tau(AQ_{m-1}) + \tau(P_m)F_{AP_m} \left( \frac{t - \tau(Q_{m-1})}{\tau(P_m)} \right) \quad \text{if } \tau(Q_{m-1}) \leq t < \tau(Q_m)$$

The assertion follows.  $\square$

We are also interested in Schur-Horn theorems in type  $II_\infty$  factors. Approximate results in this setting were recently obtained by Argerami and Massey in [1]. Let  $\mathcal{N}$  be a  $\sigma$  finite type  $II_\infty$  factor and let  $\tau$  be a faithful normal semifinite trace on  $\mathcal{M}$ . We will restrict our attention to masas  $\mathcal{A}$  that admit a normal trace preserving conditional expectation. We will refer to them as atomic masas; Such masas are generated by their finite projections. Let  $A$  be a positive trace class operator. Then, as in the case of positive operators in type  $II_1$  factors, there exists a spectral scale  $f_A$ , this time on  $[0, \infty)$  and a projection valued measure  $\mu_A$ , this time on  $[0, \infty)$  so that

$$\tau(A^n) = \int_0^\infty f_A(x)^n dm \quad \forall n, \quad A = \int_0^\infty f_A(t) d\mu_A(t)$$

Given, two trace class operators  $A$  and  $S$ , we say that  $S$  majorizes  $A$ , again written  $A \prec S$  if inequalities analogous to (2) hold. For trace class operators, it is more natural to take the closure of the unitary orbit in the trace norm than in the operator norm; We thus define

$$\mathcal{O}(S) = \overline{\{USU^* : U \in \mathcal{U}(\mathcal{M})\}}^{\|\cdot\|_1} \quad (6)$$

when  $S$  is trace class in a type  $II_\infty$  factor.

When the operators considered are not trace class, one needs to be more careful while considering majorization. As pointed out by Neumann [16], one needs to consider both the upper and lower spectral scales defined as

$$\begin{aligned}
U_A(x) &= \inf\{t \mid \tau(E_A((t, \infty))) \leq x\} \\
L_A(x) &= \sup\{t \mid \tau(E_A([0, t])) \leq x\} = -U_{-A}(x)
\end{aligned}$$

When  $A$  is trace class,  $L_A$  becomes zero. For two positive operators  $A$  and  $S$ , we say that  $S$  majorizes  $A$  if

1. We have the inequalities

$$\int_0^r U_A(x)dm \leq \int_0^r U_S(x)dm, \quad \int_0^r L_A(x)dm \geq \int_0^r L_S(x)dm, \quad 0 \leq r < \infty \quad (7)$$

2. Additionally, if there is a  $\lambda$  such that  $S - \lambda I$  is trace class, then so is  $A - \lambda I$  and  $\tau(S - \lambda I) = \tau(A - \lambda I)$ .

### 3 A local Schur-Horn theorem

Recall, see (1), that two positive operators  $A$  and  $S$  in a type  $II_1$  factor  $\mathcal{M}$  are said to be equimeasurable if  $\tau(A^n) = \tau(S^n)$  for  $n = 0, 1, \dots$ . This is equivalent to saying that the spectral measures and hence the spectral scales of  $A$  and  $S$  are identical. It is also routine to see that this is also equivalent to the existence of a sequence of unitary operators  $\{U_n\}$  so that  $\|U_n S U_n^* - A\| \rightarrow 0$ .

An example of Popa[17] shows that equimeasurable operators need not be unitarily equivalent. Let  $A$  lie inside a masa  $\mathcal{A}$ . The same example of Popa also shows that we cannot hope to even "locally" conjugate  $S$  into  $A$ , i.e, it is not possible to find a unitary  $U$  and a projection  $P$  in  $\mathcal{A}$  so that  $E(PUSU^*P) = AP$  and  $A(I - P) \prec (I - P)USU^*(I - P)$ . However, I show in proposition(3.3) that this can be accomplished whenever  $A \prec S$  but  $A$  is not equimeasurable to  $S$ .

We start off with some elementary lemmas.

**Lemma 3.1.** *Let  $P$  be a projection of trace  $\frac{1}{2}$  inside a masa  $\mathcal{A}$  in a type  $II_1$  factor  $\mathcal{M}$ . Let  $A$  be a positive operator in  $\mathcal{A}$  and let  $S$  be a positive operator in  $\mathcal{M}$  that commutes with  $P$ . With respect to the decomposition  $I = P \oplus (I - P)$ , we write(using an arbitrary partial isometry  $V$  with  $VV^* = P$  and  $V^*V = I - P$  as the matrix unit  $E_{12}$ ),*

$$A = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix} \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$$

where  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are masas in  $PMP$ , the operators  $A_1$  and  $A_2$  are in  $\mathcal{A}_1$  and  $S_1$  and  $S_2$  are in  $PMP$ . Assume that

$$\sigma(S_1) \geq \sigma(A_1) \geq \sigma(S_2) \quad (8)$$

Then, there is a unitary  $U$  so that

$$E_{\mathcal{A}}(PUSU^*P) = AP \quad \text{i.e.} \quad U \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} U^* = \begin{pmatrix} X & * \\ * & Y \end{pmatrix}$$

with  $E_{\mathcal{A}_1}(X) = A_1$ . We will automatically have that,

$$\sigma_{PMP}[(I - P)USU^*(I - P)] = \sigma_{PMP}(Y) \subset \text{conv}(\sigma(S)) = [\alpha(S_2), \|S_1\|]$$

*Proof.* Let  $H$  be the positive operator in  $\mathcal{A}_1$  determined by the formula

$$H^2 = (A_1 - E_{\mathcal{A}_1}(S_2))(E_{\mathcal{A}_1}(S_1 - S_2))^{-1}$$

The operators  $A_1$ ,  $E_{\mathcal{A}_1}(S_1)$  and  $E_{\mathcal{A}_1}(S_2)$  form a commuting set and the condition(8), it is easy to see that  $H$  is a positive contraction. Now, let  $U$  be the unitary given by

$$U = \begin{pmatrix} H & \sqrt{I - H^2} \\ \sqrt{I - H^2} & H \end{pmatrix}$$

Conjugating  $S$  by  $U$ , we have that

$$USU^* = \begin{pmatrix} HS_1H + \sqrt{I - H^2}S_2\sqrt{I - H^2} & * \\ * & \sqrt{I - H^2}S_1\sqrt{I - H^2} + HS_2H \end{pmatrix}$$

Another calculation shows that

$$\begin{aligned} E_{\mathcal{A}_1}[HS_1H + \sqrt{I - H^2}S_2\sqrt{I - H^2}] &= H^2E_{\mathcal{A}_1}(S_1) + (I - H^2)E_{\mathcal{A}_1}(S_2) \\ &= H^2(E_{\mathcal{A}_1}(S_1) - E_{\mathcal{A}_1}(S_2)) + E_{\mathcal{A}_1}(S_2) \\ &= A_1 \end{aligned} \tag{9}$$

Recall that  $\alpha(S_2)$  is the smallest point in the spectrum of  $S_2$ ,

$$\begin{aligned} \alpha(S_2) &= (I - H^2)\alpha(S_2) + H^2\alpha(S_2) \\ &< (I - H^2)\alpha(S_1) + H^2\alpha(S_2) \\ &\leq \sqrt{I - H^2}S_1\sqrt{I - H^2} + HS_2H \\ &\leq (I - H^2)\|S_1\| + H^2\|S_1\| \\ &\leq \|S_1\| \end{aligned}$$

We conclude that

$$\sigma_{PMP}(Y) = \sigma_{PMP}[\sqrt{I - H^2}S_1\sqrt{I - H^2} + HS_2H] \subset [\alpha(S_2), \|S_1\|] = \text{conv}(\sigma(S)) \tag{10}$$

□

**Lemma 3.2.** *Let  $A \prec S$  be positive operators and suppose*

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \quad S = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix}$$

where the decomposition is with respect to  $I = P \oplus Q \oplus R$  where  $P, Q$  are orthogonal projections commuting with  $A$  and  $S$  and  $R = I - P - Q$ . Suppose that

1.  $\sigma_{PMP}(A_1) \geq \sigma_{QMQ}(A_2) \geq \sigma_{RMR}(A_3)$  and  $\sigma_{PMP}(S_1) \geq \sigma_{QMQ}(S_2) \geq \sigma_{RMR}(S_3)$ ,
2.  $\tau_{PMP}(S_1 - A_1) > \frac{\tau(Q)}{\tau(P)}$ .

Then, for any positive operator  $T$  in  $QMQ$  with the same trace as  $S_2$  and so that  $\sigma_{PMP}(S_1) \geq \sigma_{QMQ}(T) \geq \sigma_{RMR}(S_3)$ , we have the majorization relation,

$$\begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \prec R := \begin{pmatrix} S_1 & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & S_3 \end{pmatrix}$$

*Proof.* Follows from the first and third assertions of lemma(2.1). □

The following proposition is the critical step in the proof of the first Schur-Horn theorem, namely theorem (4.2). It shows that the problem can be "locally" solved. More precisely,

**Proposition 3.3.** *Let  $\mathcal{A}$  be a masa in a type  $II_1$  factor  $\mathcal{M}$ . Let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be positive operators with  $A \prec S$ . Assume that  $A \notin \mathcal{O}(S)$ . Then, there is a projection  $P$  in  $\mathcal{A}$  and a unitary  $U$  in  $\mathcal{M}$  so that letting  $T = USU^*$ ,*

$$E(PTP) = AP \quad \text{and} \quad A(I - P) \prec (I - P)T(I - P)$$

*The unitary  $U$  may be chosen with the following property: Let  $Q = UPU^*$  and let  $R = P \vee Q$ . Then,  $U = RUR + (I - R)$ .*

*Proof.* Let  $f_A, f_S$  be the spectral scales of  $A, S$  respectively and let  $F_A, F_S$  be the KyFan norm functions associated to  $f$  and  $g$  respectively, see (3).

Since  $A \prec S$ , we have that

$$\int_0^r f_A(x) dm \leq \int_0^r f_S(x) dm, \quad 0 \leq r \leq 1 \quad \text{and} \quad \int_0^1 f_A(x) dm = \int_0^1 f_S(x) dm \quad (11)$$

Assume for now that  $f_A \neq f_S$  almost everywhere and that  $F_A(x) < F_S(x)$  on  $(0, 1)$ . Once we have proved the proposition under this assumption, the general case will follow using routine arguments - See the last paragraph of the proof.

Let  $I = \{x \in [0, 1] \mid f_A(x) < f_S(x)\}$  and let  $J = \{x \in [0, 1] \mid f_A(x) > f_S(x)\}$ . Pick  $0 < a < 1$  so that  $I \cap [a - \epsilon, a]$  and  $J \cap [a, a + \epsilon]$  have positive measure for every  $\epsilon > 0$ . This is possible because of the condition (11). Now, choose numbers  $b, c$  with  $0 < b < a < c < 1$  and define the number  $\alpha$

$$\alpha := \inf_{b \leq x \leq c} F_S(x) - F_A(x)$$

Since  $F_A$  and  $F_S$  are continuous and  $F_S - F_A$  is strictly positive on  $(0, 1)$ , we have that  $\alpha > 0$ .

Pick  $\epsilon < \frac{b\alpha}{2}$  and pick subsets  $X$  and  $Y$  of positive measure in  $I \cap [a - \epsilon, a]$  and  $J \cap [a, a + \epsilon]$ . Let  $L_1, L_2, L_3, L_4$  be the sets

$$L_1 = \{f_S(x) \mid x \in X\}, L_2 = \{f_A(x) \mid x \in X\}, L_3 = \{f_A(x) \mid x \in Y\}, L_4 = \{f_S(x) \mid x \in Y\}.$$

We may further arrange, by passing to subsets, if needed, that the following are satisfied:

1.  $m(X) = m(Y)$ .
2.  $L_1 > L_2 > L_3 > L_4$

See the figure below for a schematic description:

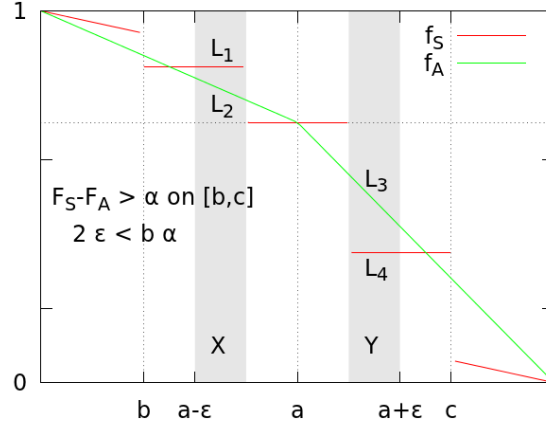


Figure 1: Illustrating proposition (3.3)

Choose a unitary  $U$  that conjugates the set  $\{\mu_S([0, a - \epsilon]), \mu_S([a - \epsilon, a + \epsilon]), \mu_S(X), \mu_S(Y)\}$  into  $\{\mu_A([0, a - \epsilon]), \mu_A([a - \epsilon, a + \epsilon]), \mu_A(X), \mu_A(Y)\}$  respectively (see (1) for the definition of  $\mu_A$  and  $\mu_S$ ). We may now write, under the decomposition

$$I = \mu_S([0, a - \epsilon]) \oplus \mu_A(X) \oplus \mu_A(Y) \oplus [\mu_S([a - \epsilon, a + \epsilon]) - \mu_A(X) - \mu_A(Y)] \oplus \mu_S([a + \epsilon, 1]),$$



$$A = \begin{pmatrix} B_1 & 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & 0 & B_2 \end{pmatrix} \quad USU^* = \begin{pmatrix} T_1 & 0 & 0 & 0 & 0 \\ 0 & S_1 & 0 & 0 & 0 \\ 0 & 0 & S_2 & 0 & 0 \\ 0 & 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & 0 & T_2 \end{pmatrix}$$

We have that

$$\sigma(S_1) > \sigma(A_1) > \sigma(A_2) > \sigma(S_2)$$

as well as

$$\sigma(B_1) \geq \sigma(A_1) \cup \sigma(A_2) \cup \sigma(A_3) \geq \sigma(B_2), \quad \sigma(T_1) \geq \sigma(S_1) \cup \sigma(S_2) \cup \sigma(S_3) \geq \sigma(T_2)$$

and further,

$$\tau(T_1 - B_1) \geq \alpha > \frac{2\epsilon}{b} > \frac{\tau[\mu_A([a - \epsilon, a + \epsilon])]}{\tau[\mu_A([0, a - \epsilon])]} \quad (12)$$

By lemma(1), there is a unitary  $V$  of the form

$$V = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}, \quad \text{so that } VUSU^*V^* = \begin{pmatrix} T_1 & 0 & 0 & 0 & 0 \\ 0 & X & * & 0 & 0 \\ 0 & * & Y & 0 & 0 \\ 0 & 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & 0 & T_2 \end{pmatrix}$$

with  $\mu_A(X) = A_1$  and

$$\sigma(T_1) \geq \sigma(X) \cup \sigma(Y) \cup \sigma(S_3) \geq \sigma(T_2) \quad (13)$$

Taking a conditional expectation,  $\tilde{E} = I \oplus E \oplus I \oplus I \oplus I$ , we have that

$$\tilde{E}(VUSU^*V^*) = R = \begin{pmatrix} T_1 & 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & Y & 0 & 0 \\ 0 & 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & 0 & T \end{pmatrix} \prec VUSU^*V^*$$

By the calculations (12) and (13), the operators  $A$  and  $R$  satisfy the hypothesis of lemma(2) and thus  $A \prec R$ , namely

$$\begin{pmatrix} B_1 & 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & 0 & B_2 \end{pmatrix} \prec \begin{pmatrix} T_1 & 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & Y & 0 & 0 \\ 0 & 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & 0 & T \end{pmatrix}$$

This implies that

$$\begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix} \prec \begin{pmatrix} T_1 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & T_2 \end{pmatrix}$$

or, in other words,

$$(I - \mu_A(X))A \prec (I - \mu_A(X))VUSU^*V^*(I - \mu_A(X))$$

Let  $P = \mu_A(X)$  and  $T = VUSU^*V^*$ . We have shown that

$$E(PTP) = AP \quad \text{and} \quad A(I - P) \prec (I - P)T(I - P),$$

which was what was required.

Now, we look at the general case, dropping the assumption that  $f \neq g$  almost everywhere and that  $F(x) < G(x)$  on  $(0, 1)$ . Let  $X \in [0, 1]$  be the maximal measurable set so that  $f|_X = g|_X$ . Pick a unitary  $U$  that conjugates  $\mu_S(X)$  into  $\mu_A(X)$ . We may write, under the decomposition  $I = \mu_A(X) \oplus \mu_A(X^c) = P \oplus Q$ ,

$$A = A_1 \oplus A_2 \quad \text{and} \quad USU^* = S_1 \oplus S_2$$

where  $A_1$  and  $S_1$  have the same spectral measure, hence  $A_1 \in \mathcal{O}(S_1)[4]$  [Theorem 5.4] and  $A_2 \prec S_2$  with the property that the spectral scales  $f_{A_2}$  and  $f_{S_2}$  inside  $QM$  satisfy  $f_{A_2} \neq f_{S_2}$  almost everywhere. Since  $A \notin \mathcal{O}(S)$ ,  $A_2$  and  $S_2$  are non-zero.

Since the KyFan norm functions are continuous, we may find two points  $\{a_1, a_2\}$  so that  $F_{A_2}(a_i) = F_{S_2}(a_i)$  and  $F_{A_2} < F_{S_2}$  on  $(a_1, a_2)$ . Pick a unitary  $V$  that conjugates  $\mu_S([a_1, a_2])$  into  $\mu_A([a_1, a_2])$  and commutes with  $\mu_A(X)$ . We may write, under the decomposition  $I = \mu_A(X) \oplus \mu_A([a_1, a_2]) \oplus [I - \mu_A(X) - \mu_A([a_1, a_2])]$ ,

$$A = A_1 \oplus A_3 \oplus A_4 \quad \text{and} \quad VUSU^*V^* = S_1 \oplus S_3 \oplus S_4$$

where  $A_3 \prec S_3$  and whose respective spectral scales are non-equal a.e. Further, the KyFan norm functions satisfy  $F_{A_3} < F_{S_3}$  on  $(0, 1)$ . Note that we also have that  $A_4 \prec S_4$ .

The proposition applies to  $(A_3, S_3)$  and yields the desired conclusion for  $A$  and  $VUSU^*V^*$ .  $\square$

We deduce the following corollary:

**Corollary 3.4.** *Let  $\mathcal{A}$  be a masa in a type  $II_1$  factor  $\mathcal{M}$ . Let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be positive operators with  $A \prec S$  and further, let  $\phi$  be an automorphism of  $\mathcal{M}$  and let  $P$  be a projection in  $\mathcal{A}$  so that  $E(P\phi(S)P) = AP$  and  $A(I - P) \prec (I - P)\phi(S)(I - P)$ . Assume that  $A(I - P) \not\approx (I - P)T(I - P)$  inside  $(I - P)\mathcal{M}(I - P)$ . Then, there are projections  $Q_1, Q_2$  in  $\mathcal{A}$  so that  $P < Q_1 < Q_2$  and an automorphism  $\psi$  of  $\mathcal{M}$  so that  $E(Q_1\psi(S)Q_1) = AQ_1$ ,  $A(I - Q_1) \prec (I - Q_1)\psi(S)(I - Q_1)$  and*

$$P\phi(\cdot)P = P\psi(\cdot)P, \quad (I - Q_2)\phi(\cdot)(I - Q_2) = (I - Q_2)\psi(\cdot)(I - Q_2)$$

Further,  $\tau(Q_2 - Q_1) \leq \tau(Q_1 - P)$ .

*Proof.* Using proposition(3.3), choose a unitary  $U$  in  $(I - P)\mathcal{M}(I - P)$  and a projection  $Q_1 > P$  so that letting  $\psi = \text{Ad}(P \oplus U) \circ \phi$ , we have that

$$E(Q_1\psi(S)Q_1) = AQ_1, \quad \text{and} \quad A(I - Q_1) \prec (I - Q_1)\psi(S)(I - Q_1)$$

Clearly,  $P\phi(\cdot)P = P\psi(\cdot)P$ . Also, let  $Q_2 = P + Q_1 \vee (P \oplus U)Q_1(P \oplus U)^* - Q_1$ . Then,  $(I - Q_2)U(I - Q_2) = I - Q_2$  which yields that  $(I - Q_2)\phi(\cdot)(I - Q_2) = (I - Q_2)\psi(\cdot)(I - Q_2)$ . Finally, we have that

$$\begin{aligned} \tau(Q_2 - Q_1) &= \tau(Q_1 \vee (P \oplus U)Q_1(P \oplus U)^* - Q_1) \\ &\leq 2\tau(Q_1) - \tau(Q_1 \wedge (P \oplus U)Q_1(P \oplus U)^*) - \tau(Q_1) \\ &\leq 2\tau(Q_1) - \tau(P) - \tau(Q_1) \\ &\leq \tau(Q_1 - P). \end{aligned}$$

We are done.  $\square$

## 4 Diagonals of positive operators in type $II_1$ factors

We deduce a Schur-Horn theorem in type  $II_1$  factors from corollary(3.4) using an induction argument.

**Theorem 4.1.** *Let  $\mathcal{A}$  be a masa in a type  $II_1$  factor  $\mathcal{M}$ . If  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  are positive operators with  $A \prec S$ . Then, there is an automorphism  $\phi$  of  $\mathcal{M}$  and a projection  $P$  in  $\mathcal{A}$  such that*

$$E(P\phi(S)P) = AP \quad \text{and} \quad (I - P)\phi(S)(I - P) \approx A(I - P).$$

*Proof.* If  $A \in \mathcal{O}(S)$ , there is nothing to prove; Just set  $P$  to be zero and the autmorphisms to be the identity. Let us therefore assume that  $A \notin \mathcal{O}(S)$

Let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be positive operators so that  $A \prec S$ . Let  $\mathcal{X}$  be the collection of all tuples  $\{\phi, P\}$  where  $\phi \in \text{Aut}(\mathcal{M})$  such that  $A \prec \phi(S)$  and  $P$  is a projection with  $E(P\phi(S)P) = AP$  and  $A(I - P) \prec (I - P)\phi(S)(I - P)$ . Define an ordering  $\leq$  on the set  $\mathcal{X}$  by  $(\phi_1, P_1) \leq (\phi_2, P_2)$  if

1.  $P_1 \leq P_2$ , i.e  $P_1 P_2 = P_1$ ,
2.  $P_1 \phi_1(\cdot) P_1 = P_1 \phi_2(\cdot) P_1$  and
3. There is a projection  $Q$ , with  $Q > P_2$  and satisfying  $\tau(Q - P_2) \leq \tau(P_2 - P_1)$  so that  $(I - Q)\phi_1(\cdot)(I - Q) = (I - Q)\phi_2(\cdot)(I - Q)$ .

The set  $\mathcal{X}$  with the given ordering is a poset. To see this, suppose  $(\phi_1, P_1) \leq (\phi_2, P_2)$  and  $(\phi_2, P_2) \leq (\phi_3, P_3)$ ; While showing that  $(\phi_1, P_1) \leq (\phi_3, P_3)$ , properties (1) and (2) are immediate. Let  $Q_1, Q_2$  be the projections that ensure condition (3) in the inequalities  $(\phi_1, P_1) \leq (\phi_2, P_2)$  and  $(\phi_2, P_2) \leq (\phi_3, P_3)$  respectively. Take  $Q_3 = Q_1 \vee Q_2$ . It is clear that  $(I - Q_3)\phi_3(\cdot)(I - Q_3) = (I - Q_3)\phi_1(\cdot)(I - Q_3)$ . Now,

$$\begin{aligned} \tau(Q_3 - P_3) &= \tau(Q_1 \vee Q_2) - \tau(P_3) \\ &= \tau(Q_1) + \tau(Q_2) - \tau(Q_1 \wedge Q_2) - \tau(P_3) \\ &\leq 2\tau(P_2) - \tau(P_1) + 2\tau(P_3) - \tau(P_2) - \tau(P_2) - \tau(P_3) \\ &= \tau(P_3 - P_1) \end{aligned} \tag{14}$$

Thus,  $(\mathcal{X}, \leq)$  is a poset.

Let  $T$  be an arbitrary operator in  $\mathcal{M}$  and suppose  $(\phi_1, P_1) \leq (\phi_2, P_2)$ . Combining (2) and (3), we have that

$$(\phi_1 - \phi_2)(T) = (Q - P_1)((\phi_1 - \phi_2)(T)) + ((\phi_1 - \phi_2)(T))(Q - P_1) - (Q - P_1)((\phi_1 - \phi_2)(T))(Q - P_1)$$

We now estimate the norm of the difference,

$$\begin{aligned} \|(\phi_1 - \phi_2)(T)\|_2 &\leq 2\|(Q - P_1)(\phi_1 - \phi_2)(T)\|_2 + \|(Q - P_1)(\phi_1 - \phi_2)(T)(Q - P_1)\|_2 \\ &\leq 3\|Q - P_1\|_2\|\phi_1(T) - \phi_2(T)\| \\ &\leq 12\tau(P_2 - P_1)\|T\| \end{aligned} \tag{15}$$

Let  $\{(\phi_\alpha, P_\alpha)\}_{\alpha \in I}$  be a chain in  $\mathcal{X}$ . Since the projections  $P_\alpha$  are increasing, they have a strong operator limit, which we denote by  $P$ . Since the net  $\{P_\alpha\}$  converges in the  $\|\cdot\|_2$  norm, (15) yields that the net of automorphisms  $\{\phi_\alpha\}$  must converge as well in the point 2 norm. Denote the limit automorphism by  $\phi$ . Now,

$$E(P\phi(S)P) = \lim_{\text{SOT}} E(P_\alpha \phi_\alpha(S) P_\alpha) = \lim_{\text{SOT}} AP_\alpha = AP$$

Concerning majorization,  $A(I - P_\alpha) \prec (I - P_\alpha)\phi_\alpha(S)(I - P_\alpha)$  for every  $\alpha$  and hence, passing to the strong operator limit,

$$A(I - P) \prec (I - P)\phi(S)(I - P).$$

We now show that  $(\phi_\alpha, P_\alpha) \leq (\phi, P)$ . We have only to show this in the case the set  $\{\beta \in I : \beta \geq \alpha\}$  equals  $\mathbb{N}$ . So let  $\{(\phi_n, P_n)\}_{n \in \mathbb{N}}$  be a chain and suppose that for  $n = 1, 2, \dots$ , the projection  $Q_n$  controls  $(\phi_{n+1}, P_{n+1})$  with respect to  $(\phi_n, P_n)$  by satisfying condition (3). We will show that  $(\phi_1, P_1) \leq (\phi, P)$  where  $P = \vee_n P_n$ ; Set  $Q = \vee_n Q_n$ .

It is easy to see that  $(I - Q)\phi_1(\cdot)(I - Q) = (I - Q)\phi(\cdot)(I - Q)$ . If we show that  $\tau(Q) - \tau(P) \leq \tau(P) - \tau(P_1)$ , we will be done. A calculation similar to (14) shows us that

$$\tau(\vee_1^n Q_m) - \tau(P_n) \leq \tau(P_n) - \tau(P_1) \quad (16)$$

Taking the limit as  $n \rightarrow \infty$ , we get that  $\tau(Q) - \tau(P) \leq \tau(P) - \tau(P_1)$ . Returning to the main proof, we conclude that  $(\phi, P)$  is in  $\mathcal{X}$ . We have just shown that every chain in  $\mathcal{X}$  has an upper bound. By Zorn's lemma, there is a maximal element in  $\mathcal{X}$ . Call this element  $(\phi, P)$ . If  $(I - P)\phi(S)(I - P)$  is not equimeasurable with  $A(I - P)$ , this would contradict maximality. The theorem follows.

By corollary(3.4), we have that  $A(I - P)$  is in  $\mathcal{O}((I - P)\mathcal{M}(I - P))$  inside  $(I - P)\mathcal{M}(I - P)$ . We can find a masa  $\mathcal{A}_1$  inside  $(I - P)\mathcal{M}(I - P)$  so that  $(I - P)\mathcal{M}(I - P)$  lies inside  $\mathcal{A}_1$ . We conclude that the conditional expectation of  $T$  onto  $\mathcal{AP} \oplus \mathcal{A}_1$  is equimeasurable with  $A$ .  $\square$

We now prove the first of the two generalizations of the Schur-Horn theorem to type  $II_1$  factors. We repeat the statement of the theorem for the convenience of the reader.

**Theorem 4.2.** *[The Schur-Horn theorem in type  $II_1$  factors I] Let  $\mathcal{M}$  be a type  $II_1$  factor and let  $A, S \in \mathcal{M}$  be positive operators with  $A \prec S$ . Then, there is some masa  $\mathcal{A}$  in  $\mathcal{M}$  such that*

$$E_{\mathcal{A}}(S) \approx A.$$

*Proof.* Choose a masa  $\mathcal{A}_1$  such that  $A$  belongs to  $\mathcal{A}_1$ . Theorem (4.1) yields that there is an automorphism  $\phi$  in  $\text{Aut}(\mathcal{M})$  and a projection  $P$  in  $\mathcal{A}$  such that

$$E(P\phi(S)P) = AP \quad \text{and} \quad (I - P)\phi(S)(I - P) \approx A(I - P).$$

Choose a masa  $\tilde{\mathcal{A}}$  in  $(I - P)\mathcal{M}(I - P)$  that contains  $(I - P)\phi(S)(I - P)$ . Then, we have that

$$E_{\mathcal{A}_1 P \oplus \tilde{\mathcal{A}}}(\phi(S)) = AP \oplus (I - P)\phi(S)(I - P) \quad (17)$$

Note that we have that  $A \approx AP \oplus (I - P)\phi(S)(I - P)$ . Let  $\mathcal{A}$  be the masa  $\phi^{-1}(\mathcal{A}_1 P \oplus \tilde{\mathcal{A}})$ . We then get by applying the automorphism  $\phi^{-1}$  to the equation (17) that

$$E_{\mathcal{A}}(S) = \phi^{-1}(AP \oplus (I - P)\phi(S)(I - P))$$

Since  $\phi^{-1}(AP \oplus (I - P)\phi(S)(I - P)) \approx AP \oplus (I - P)\phi(S)(I - P) \approx A$ , we are done.  $\square$

**Remark 4.3.** *The above theorem, as remarked in the introduction, is a natural generalization of the Schur-Horn theorem to type  $II_1$  factors. This generalization does not directly imply the alternative conjecture of Arveson and Kadison from [4]. However, when  $A$  has finite spectrum, it is routine to see that the Arveson-Kadison conjecture can be deduced from above. We prove this conjecture in full in the next section.*

## 5 Proof of the Arveson-Kadison conjecture

We now turn to the second natural generalization of the Schur-Horn theorem. The theorem of the last section characterizes the spectral distributions of operators that arise as the “diagonal” of a given positive operator  $S$ . On the other hand, the conjecture of Arveson and Kadison complements the above-mentioned theorem by characterizing the spectral distributions of operators which have a prescribed diagonal  $A$ .

The main result in this section is the proof of theorem (5.6). We first prove a couple of lemmas.

**Lemma 5.1.** *Let  $\mathcal{A}$  be a masa in a type  $II_1$  factor  $\mathcal{M}$  and let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be two positive operators commuting with a projection  $P$  of trace  $\frac{1}{2}$  in  $\mathcal{A}$ , written with respect to the decomposition  $I = P \oplus I - P$ ,*

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

*where  $f_{S_1} - f_{A_1} > 0$  and  $\sigma(A_1) \geq \sigma(A_2)$  and also  $A_2 \approx S_2$ . Then, for any  $\delta > 0$ , there is a unitary  $U$  and projections  $R_1 \leq P$  and  $R_2 \leq I - P$ , both in  $\mathcal{A}$ , with  $\tau(R_i) > \frac{1}{2} - \delta$  for  $i = 1, 2$  such that  $E_{\mathcal{A}R_1}(R_1 U S U^* R_1) = A_1 R_1$  and  $f_{R_2 U S U^* R_2} - f_{A_2 R_2} > 0$ .*

*Proof.* Let  $\delta > 0$  be fixed. It is easy to see (using the fact that the spectral scales  $f_A$  and  $f_S$  are right continuous) that we may find a natural number  $k$ , a number  $\epsilon > 0$  and disjoint intervals  $(a_1, a_1 + \epsilon), \dots, (a_{k+1}, a_{k+1} + \epsilon)$  in  $[0, 1]$  with  $a_1 < a_2 \dots < a_k < a_{k+1}$  such that

1.  $k\epsilon > 1 - 2\delta$ . And,
2.  $f_{S_1}(x) - f_{A_1}(y) > 0$  for  $x, y \in (a_i, a_i + \epsilon)$  for  $i = 1, \dots, k+1$ .

For  $i = 1, \dots, k+1$ , denote the projections  $\mu_{A_1}((a_i, a_i + \epsilon))$  by  $P_i$  and  $\mu_{A_2}((a_i, a_i + \epsilon))$  by  $Q_i$ . Pick a unitary  $U_1$  in  $PMP$  that conjugates  $\mu_{S_1}((a_i, a_i + \epsilon))$  onto  $P_i$  and a unitary  $U_2$  in  $(I - P)\mathcal{M}(I - P)$  that conjugates  $\mu_{S_2}((a_i, a_i + \epsilon))$  onto  $Q_i$  for  $i = 1, \dots, k+1$ . Let  $U := U_1 \oplus U_2$  and let  $T := U S U^*$ . Then,  $T$  commutes with the projections  $P_i$  and  $Q_i$ .

For  $i = 1, \dots, k+1$ , let  $X_i := T(P_i \oplus Q_i)$  and  $Y_i := A(P_i \oplus Q_i)$ . For each  $i = 1, \dots, k+1$ , the pair of operators  $A(P_i \oplus Q_i)$  and  $T(P_i \oplus Q_i)$  inside  $(P_i \oplus Q_i)\mathcal{M}(P_i \oplus Q_i)$  satisfy the hypothesis of lemma(3.1) and we may thus find unitaries  $V_i$  in  $(P_i \oplus Q_i)\mathcal{M}(P_i \oplus Q_i)$  such that

$$E_{\mathcal{A}P_i}(P_i V_i X_i V_i^* P_i) = Y_i P_i, \quad \sigma(Q_i V_i X_i V_i^* Q_i) \subset \text{conv}(\sigma(X_i)) \quad (18)$$

Let  $W$  be a unitary in  $(I - P)\mathcal{M}(I - P)$  that conjugates  $Q_{i+1}$  onto  $Q_i$  for  $i = 2, \dots, k+1$ . The second fact above gives us that

$$\sigma(Q_{i+1} W V_i X_i V_i^* W^* Q_{i+1}) \geq \sigma(Q_{i+1} A Q_{i+1}) \quad (19)$$

Let  $U$  be the unitary  $(P \oplus W)(I - \sum_{i=1}^{k+1} (P_i \oplus Q_i) + \sum_{i=1}^{k+1} V_i)$ . Also, let  $R_1 = \sum_{i=1}^{k+1} P_i$  and  $R_2 = \sum_{i=1}^k Q_i$ . The two facts, (18) and (19) give us that

$$E_{\mathcal{A}Q_1}(R_1 U S U^* R_1) = A_1 R_1, \quad f_{R_2 U S U^* R_2} - f_{A_2 R_2} > 0$$

Finally, we have that  $\tau(R_2) = k\epsilon$  and  $\tau(R_1) = (k+1)\epsilon$  and both are greater than  $1 - 2\delta$ . We are done.  $\square$

**Lemma 5.2.** *Let  $\mathcal{A}$  be a masa in a type  $II_1$  factor  $\mathcal{M}$  and let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be two positive operators commuting with a projection  $P$  in  $\mathcal{A}$ , written with respect to the decomposition  $I = P \oplus I - P$ ,*

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

*Assume  $f_{S_1} - f_{A_1} > 0$  and  $\sigma(A_1) \geq \sigma(A_2)$  and  $A_2 \approx S_2$ . Then, there is a projection  $Q$  in  $\mathcal{A}$  with  $\tau(Q) > 1 - 2\tau(P)$  and a unitary  $U$  such that*

$$E_{\mathcal{A}}(Q U S U^* Q) = A Q$$

*Proof.* We may assume that  $\tau(P) \leq \frac{1}{2}$  for otherwise there is nothing to prove. Let  $k$  be the natural number such that  $(k+1)\tau(P) \leq 1 < (k+2)\tau(P)$ ; Note that  $k > 1$ . Now, choose a  $\delta$  such that  $0 < \frac{k(k+1)\delta}{2} < (k+2)\tau(P) - 1$ . Next, define the sequence of numbers  $\{a_1, \dots, a_k\}$  using the following prescription:  $a_1$  is such that  $\tau(\mu_{A_2}((a_1, 1))) = \tau(P)$  and for  $i = 2, \dots, k$ , the number  $a_i$  is such that  $\tau(\mu_{A_2}((a_i, a_{i-1}))) = \tau(P) - (i-1)\delta$ .

Now, define the sequence of projections  $P_1, \dots, P_k$  by  $P_1 = \mu_{A_2}((a_1, 1))$  and then for  $i = 2, \dots, k$ , define  $P_i = \mu_{A_2}((a_i, a_{i-1}))$ . Pick a unitary  $V$  that conjugates  $\mu_{S_2}(a_1, 1)$  onto  $\mu_{A_2}(a_1, 1)$  as well as each of  $\mu_{S_2}((a_i, a_{i-1}))$  onto  $\mu_{A_2}((a_i, a_{i-1}))$  for  $i = 1, \dots, k$  and which commutes with  $P$ . We then have,

$$A = \begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & A_{21} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & A_{2k} & 0 \\ 0 & 0 & 0 & 0 & A_{2k+1} \end{pmatrix}, \quad VSV^* = \begin{pmatrix} S_{11} & 0 & 0 & 0 & 0 \\ 0 & S_{21} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & S_{2k} & 0 \\ 0 & 0 & 0 & 0 & S_{2k+1} \end{pmatrix}$$

where  $S_{2i} \approx A_{2i}$  for  $i = 1, \dots, k$  and  $\sigma(S_{21}) \geq \dots \geq \sigma(S_{2k}) \geq \sigma(S_{2k+1})$ . Also,  $f_{S_{11}} - f_{A_{11}} > 0$ .

Applying lemma(5.1) to  $A_{11} \oplus A_{21}$  and  $S_{11} \oplus S_{21}$  inside  $(P \oplus P_1)\mathcal{M}(P \oplus P_1)$ , we conclude that we may find a unitary  $U_1$  commuting with  $I - P - P_1$  and projections  $Q_1$  and  $R_1$  of trace  $\tau(P) - \delta$  in  $\mathcal{A}$  with  $Q_1 \leq P$  and  $R_1 \leq P_1$  such that letting  $T_1 = U_1 S U_1^*$ ,  $E_{\mathcal{A}}(Q_1 T_1 Q_1) = A Q_1$  and further,  $f_{R_1 T_1 R_1} - f_{A_{21} R_1} > 0$ .

Inductively, for  $i = 1, \dots, k-1$ , do the following: Note that  $f_{R_i T_i R_i} - f_{A_{2i} R_i} > 0$  and apply lemma(5.1) to  $A_{2i} R_i \oplus A_{2i+1}$  and  $T_i \oplus S_{2i+1}$  inside  $(R_i \oplus P_{i+1})\mathcal{M}(R_i \oplus P_{i+1})$ . The lemma yields a unitary  $U_{i+1}$  commuting with  $I - R_i - P_{i+1}$  and projections  $Q_{i+1}$  and  $R_{i+1}$  of trace  $\tau(P) - i\delta$  in  $\mathcal{A}$  with  $Q_{i+1} \leq P_i$  and  $R_{i+1} \leq P_{i+1}$  such that letting  $T_{i+1} = U_{i+1} T_i U_{i+1}^*$ , we have  $E_{\mathcal{A}}(Q_{i+1} T_{i+1} Q_{i+1}) = A Q_{i+1}$  and further, we have  $f_{R_{i+1} T_{i+1} R_{i+1}} - f_{A_{2i+1} R_{i+1}} > 0$ .

Putting it all together, we have that

$$E_{\mathcal{A}}(QUSU^*Q) = AQ$$

where  $Q = Q_1 \oplus Q_2 \cdots Q_k$ . We have that  $\tau(Q_i) = \tau(P) - i\delta$  and thus,

$$\tau(Q) = \sum_{i=1}^k \tau(P) - i\delta \tag{20}$$

$$= k\tau(P) - \frac{k(k+1)}{2}\delta \tag{21}$$

$$> 1 - 2\tau(P) \tag{22}$$

□

We now turn to the main theorem of the paper, the proof of the conjecture (5.6) of Arveson and Kadison in [4]. We start off with some preliminary remarks. Let  $f_A$  and  $f_S$  be the spectral scales of  $A$  and  $S$  respectively. Define

$$\mathcal{E} := \{x \in (0, 1) : f_A(x) = f_S(x)\}$$

Choose a unitary that conjugates  $\mu_S(\mathcal{E})$  onto  $\mu_A(\mathcal{E})$ . With respect to the decomposition  $I = \mu_A(\mathcal{E}) \oplus \mu_A(\mathcal{E}^c) = (I - P) \oplus P$ , we may write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad USU^* = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$$

Then,  $A_1 \approx S_1$  and  $A_2 \prec S_2$  inside  $PMP$ . It is now easy to see that if we can prove the theorem for  $A_2$  and  $S_2$  inside  $PMP$ , the result for  $A$  and  $S$  inside  $\mathcal{M}$  would follow. We may therefore assume that  $f_A \neq f_S$  almost everywhere on  $[0, 1]$ .

Let  $F_A$  and  $F_S$  be the Ky Fan norm functions. The relation  $A \prec S$  gives us that  $F_A \leq F_S$  on  $[0, 1]$ . Define

$$\mathcal{F} := \{x \in (0, 1) : F_A(x) = F_S(x)\}$$

Since we assume that  $f_A \neq f_S$  almost everywhere on  $[0, 1]$ ,  $\mathcal{F}$  cannot contain any intervals. We may write  $\mathcal{F}^c$  as a union of disjoint intervals  $\{I_\alpha\}$ ; Pick a unitary  $U$  that conjugates  $\mu_S(I_\alpha)$  onto  $\mu_A(I_\alpha)$  for every  $\alpha$ . Then,

$$A = \sum_{\alpha} A \mu_A(I_\alpha) \quad \text{and} \quad USU^* = \sum_{\alpha} USU^* \mu_A(I_\alpha)$$

where  $A \mu_A(I_\alpha) \prec USU^* \mu_A(I_\alpha)$  and further the corresponding KyFan norm functions are strictly positive on  $(0, 1)$  for every  $\alpha$ . It is routine to see that if we can solve the problem for every  $\alpha$ , the general theorem follows. Therefore, we may assume, additionally to  $f_A \neq f_S$  a.e. on  $[0, 1]$ , that  $F_A < F_S$  on  $(0, 1)$ .

We use the following notation:

$$A \prec_w S \quad \text{if} \quad A \prec_w S, \quad F_A < F_S \text{ on } (0, 1) \quad (23)$$

If we have that  $A \prec_w S$  and also  $\tau(A) = \tau(S)$ , we say that  $A \prec S$ .

We need one last fact, a lemma analogous to lemma (3.2).

**Lemma 5.3.** *Let  $A$  and  $S$  be positive operators and suppose, with respect to  $I = P \oplus Q \oplus R$ , we have*

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \quad S = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix}$$

where  $\sigma(A_1) \geq \sigma(A_2) \cup \sigma(A_3)$  and  $\sigma(S_1) \geq \sigma(S_2) \cup \sigma(S_3)$ . Suppose that  $A_1 \prec_w S_1$  and

$$\tau_{PM}(S_1 - A_1) + \frac{\tau(Q)}{\tau(P)} L(A_2, S_2) - \frac{\tau(R)}{\tau(P)} \geq 0$$

Then,  $A \prec_w S$ . If, in addition, we have that  $A_1 \prec_w S_1$ , then,  $A \prec_w S$ .

*Proof.* Let  $\tilde{A}$  and  $\tilde{S}$  be the operators  $A(I - P)$  and  $S(I - P)$  in  $(I - P)\mathcal{M}(I - P)$ . Then the assertion (1) in lemma(2.1) says that

$$L(\tilde{A}, \tilde{S}) \geq \frac{\tau(Q)}{\tau(Q + R)} L(A_2, S_2) + \frac{\tau(R)}{\tau(Q + R)} L(A_3, S_3)$$

and hence

$$\frac{\tau(I - P)}{\tau(P)} L(\tilde{A}, \tilde{S}) \geq \frac{\tau(Q)}{\tau(P)} L(A_2, S_2) + \frac{\tau(R)}{\tau(P)} L(A_3, S_3)$$

Using the crude estimate that  $L(A_3, S_3) \geq -1$ , we conclude that

$$\begin{aligned} \tau_{PM}(S_1 - A_1) + \frac{\tau(I - P)}{\tau(P)} L(\tilde{A}, \tilde{S}) &\geq \tau_{PM}(S_1 - A_1) + \frac{\tau(Q)}{\tau(P)} L(A_2, S_2) + \frac{\tau(R)}{\tau(P)} L(A_3, S_3) \\ &\geq \tau_{PM}(S_1 - A_1) + \frac{\tau(Q)}{\tau(P)} L(A_2, S_2) - \frac{\tau(R)}{\tau(P)} \\ &\geq 0 \end{aligned}$$

The assertion (1) in lemma (2.1) now applies and we conclude that

$$A \prec_w S$$

The proof of the second assertion is similar and we omit it.  $\square$

**Proposition 5.4.** *Let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be positive operators with  $A \lesssim S$ . Then, there is a projection  $P$  in  $\mathcal{A}$  with  $\tau(P) \geq \frac{1}{2}$  and an automorphism  $\phi$  of  $\mathcal{M}$  such that  $E(P\phi(S)P) = AP$  and  $A(I - P) \lesssim (I - P)\phi(S)(I - P)$ .*

*Proof of proposition (5.4).* We may assume that both  $A$  and  $S$  belong to  $\mathcal{A}$ , are contractions and that further, for a projection valued measure as in (1)

$$A = \int f_A(t) d\mu_A(t), \quad S = \int f_S(t) d\mu_A(t)$$

By assumption, we have that  $F_S > F_A$  on  $(0, 1)$ . Choose numbers  $0 < a < \frac{1}{8} < \frac{7}{8} < b < 1$  such that

$$F_S(x) - F_A(x) > F_S(a) - F_A(a) = F_S(b) - F_A(b), \quad a < x < b.$$

With respect to the decomposition  $I = \mu_A((0, a)) \oplus \mu_A((a, b)) \oplus \mu_A((b, 1))$ , we may write

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_3 & 0 \\ 0 & 0 & A_2 \end{pmatrix} \quad S = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_3 & 0 \\ 0 & 0 & S_2 \end{pmatrix}$$

Let  $Q = \mu_A((a, b))$ ; Note that  $\tau(Q) = b - a > \frac{7}{8} - \frac{1}{8} = \frac{3}{4}$ . We have that

$$A_3 \prec S_3 \text{ inside } Q\mathcal{M}Q, \quad A_1 \oplus A_2 \prec S_1 \oplus S_2 \text{ inside } (I - Q)\mathcal{M}(I - Q).$$

Note further that

$$\sigma(A_1) \geq \sigma(A_3) \geq \sigma(A_2), \quad \sigma(S_1) \geq \sigma(S_3) \geq \sigma(S_2).$$

Apply theorem (4.2) to the pair  $A_3$  and  $S_3$  inside  $Q\mathcal{M}Q$ . We get an automorphism  $\Phi = I \oplus \phi \oplus I$  of  $\mathcal{M}$  so that with respect to the decomposition  $I = \mu_A((0, a)) \oplus Q \oplus \mu_A((b, 1))$ , we have

$$\Phi(S) = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & S_2 \end{pmatrix}, \quad T \approx A_3 \tag{24}$$

Choose a number  $c$  in  $(0, a)$  such that  $f_S(c) > f_A(c)$ . Recall that the spectral scales are right continuous. Thus, we may find an interval  $I = [d, c]$  such that  $f_S > f_A$  on  $I$ . Now, choose a  $\delta > 0$  satisfying

$$\delta < \frac{\tau(Q)}{3}, \quad f_S > f_A \text{ on } [c - \delta, c], \tag{25}$$

as well as

$$F_S - F_A > 3\delta \text{ on } [c - \delta, a]. \tag{26}$$

Choose a unitary  $V_1$  that is the identity on  $I - \mu_A((0, a))$  that conjugates the set

$$\{\mu_{\Phi(S)}((0, c - \delta)), \mu_{\Phi(S)}((c - \delta, c)), \mu_{\Phi(S)}((c, a))\}$$

onto the set

$$\{\mu_A((0, c - \delta)), \mu_A((c - \delta, c)), \mu_A((c, a))\}$$



Now, with respect to the decomposition  $I = \mu_A((0, c - \delta)) \oplus \mu_A(c - \delta, c) \oplus \mu_A((c, a)) \oplus Q \oplus \mu_A(b, 1)$ , we may write

$$A = \begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 & 0 \\ 0 & 0 & A_{13} & 0 & 0 \\ 0 & 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & 0 & A_2 \end{pmatrix}, \quad V\Phi(S)V^* = \begin{pmatrix} S_{11} & 0 & 0 & 0 & 0 \\ 0 & S_{12} & 0 & 0 & 0 \\ 0 & 0 & S_{13} & 0 & 0 \\ 0 & 0 & 0 & T & 0 \\ 0 & 0 & 0 & 0 & S_2 \end{pmatrix}$$

Let  $Q_1$  be the projection  $\mu_A(c - \delta, c)$ ; We have that  $\tau(Q_1) = \delta$ . Compressing to  $(Q_1 \oplus Q)\mathcal{M}(Q_1 \oplus Q)$ , and letting  $B = A(Q_1 \oplus Q)$  and  $R = V\Phi(S)V^*(Q_1 \oplus Q)$ , we have

$$B = \begin{pmatrix} A_{12} & 0 \\ 0 & A_3 \end{pmatrix}, \quad R = \begin{pmatrix} S_{12} & 0 \\ 0 & T \end{pmatrix}$$

Note that we have the following,

$$A_3 \approx T, \quad \tau(Q_1) < \frac{\tau(Q)}{3}, \quad f_{S_{12}} > f_{A_{12}}, \quad \sigma(A_{12}) \geq \sigma(A_3) \quad (27)$$

The first assertion follows from (24) and the second and third from (25).

Applying lemma(5.2) to  $B$  and  $R$ , we get a projection  $P$  in  $\mathcal{A}$  with  $P \leq Q_1 \oplus Q$  and a unitary  $U$  that is the identity on  $I - Q_1 - Q$  such that

$$E_{\mathcal{A}}(PUV\Phi(S)V^*U^*P) = AP$$

and also,

$$\tau(P) > \tau(Q_1 + Q) - 2\tau(Q_1) > \frac{2\tau(Q)}{3} > \frac{1}{2}$$

We have with respect to  $I = \mu_A((0, c - \delta)) \oplus \mu_A((c, a)) \oplus P \oplus Q_1 + Q - P \oplus \mu_A(b, 1)$ ,

$$A = \begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & A_{13} & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 \\ 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & A_2 \end{pmatrix}, \quad UV\Phi(S)V^*U^* = \begin{pmatrix} S_{11} & 0 & 0 & 0 & 0 \\ 0 & S_{13} & 0 & 0 & 0 \\ 0 & 0 & X & * & 0 \\ 0 & 0 & * & Y & 0 \\ 0 & 0 & 0 & 0 & S_2 \end{pmatrix}$$

where  $E_{\mathcal{A}Q_1}(X) = B$  and also,  $\tau(Q_1 + Q - P) \leq 2\tau(Q_1 + Q)\tau(Q_1) < 2\delta$ .

In the last part of the proof, I show that  $A(I - P) \lesssim (I - P)UV\Phi(S)V^*U^*(I - P)$ . For brevity, denote by  $T$  the operator  $UV\Phi(S)V^*U^*$  and by  $R$  the projection  $I - \mu_A((b, 1))$ .

We have, inside  $R\mathcal{M}R$ , with respect to  $I = \mu_A((0, c - \delta)) \oplus \mu_A((c, a)) \oplus Q_1 + Q - P$ ,

$$AR = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{13} & 0 \\ 0 & 0 & C \end{pmatrix}, \quad RTR = \begin{pmatrix} S_{11} & 0 & 0 \\ 0 & S_{13} & 0 \\ 0 & 0 & Y \end{pmatrix}$$

Let us now unpack condition (26):  $F_S - F_A > 3\delta$  on  $[c - \delta, a]$ . This implies, in particular, that  $(c - \delta)\tau(S_{11} - A_{11}) + \delta\tau(S_{12} - A_{12}) + (a - c)L(A_{13}, S_{13}) > 3\delta$ . We assumed in the first line of the proof that  $A$  and  $S$  are contractions and thus,

$$(c - \delta)\tau(S_{11} - A_{11}) + (a - c)L(A_{13}, S_{13}) > 2\delta$$

Lemma (5.3) now applies and we conclude that  $AR \lesssim_w RTR$ . It is now routine to see that  $A(I - P) \lesssim (I - P)R(I - P)$ . □

**Corollary 5.5.** *Let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be positive operators and suppose we have a projection  $P$  in  $\mathcal{A}$  and an automorphism  $\phi$  of  $\mathcal{M}$  such that*

$$E(P\phi(S)P) = AP, \quad A(I - P) \lesssim (I - P)\phi(S)(I - P).$$

*Then, there is a projection  $Q$  in  $\mathcal{A}$  such that  $Q > P$  with  $\tau(I - Q) \leq \frac{\tau(I - P)}{2}$  and an automorphism  $\psi$  of  $\mathcal{M}$  such that*

$$P\psi(\cdot)P = P\psi(\cdot)P, \quad E(Q\psi(S)Q) = AQ, \quad A(I - Q) \lesssim (I - Q)\psi(S)(I - Q).$$

This is proved in the same way that corollary(3.4) is deduced from proposition (3.3) and we omit the proof.

Corollary (5.5) will imply the main Schur-Horn theorem. The passage from a partial solution to the full solution of the problem can be done exactly as in the proof of theorem (4.2).

**Theorem 5.6.** *[The Schur-Horn theorem in type  $II_1$  factors II] Let  $\mathcal{A}$  be a masa in a type  $II_1$  factor  $\mathcal{M}$ . If  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  are positive operators with  $A \prec S$ . Then, there is an element  $T \in \mathcal{O}(S)$  such that*

$$E(T) = A$$

*Proof.* Assume first that  $A \lesssim S$ . Using proposition (5.4) and corollary (5.5), we may pick a sequence of projections  $\{P_n\}$  in  $\mathcal{A}$  and a sequence of automorphisms  $\{\phi_n\}$  of  $\mathcal{M}$  such that

$$E(P_n\psi(S)P_n) = AP_n, \quad A(I - P_n) \lesssim (I - P_n)\phi_n(S)(I - P_n)$$

as well as

$$P_n\phi_{n+1}(\cdot)P_n = P_n\phi_n(\cdot)P_n$$

We may choose the  $P_n$  so that

$$\tau(P_1) \geq \frac{1}{2}, \quad \tau(I - P_{n+1}) \leq \frac{\tau(I - P_n)}{2}$$

yielding that  $\tau(P_n) \geq 1 - \frac{1}{2^n}$ . It is now routine to see that the automorphisms  $\phi_n$  converge in the pointwise strong operator topology to an automorphism that we denote  $\phi$  and we have that

$$E(\phi(S)) = A.$$

For the general case, as in the discussion preceding proposition (5.4), we can find a unitary  $U$  and a projection  $Q$  in  $\mathcal{A}$  so that with respect to  $I = I - Q \oplus Q$ ,

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$$

where  $A_1 \approx S_1$  and  $A_2 \lesssim S_2$ . Then, there is an automorphism  $\phi$  of  $Q\mathcal{M}Q$ , such that  $E_{\mathcal{A}Q}(\phi(S_2)) = AQ$ . It is routine to see that the operator  $T$  defined by

$$T := \begin{pmatrix} A_1 & 0 \\ 0 & \phi(S_2) \end{pmatrix}$$

is such that  $E(T) = A$  and that  $T$  is in  $\mathcal{O}(S)$ . □

**Remark 5.7.** *Reading through the proof of theorem(5.6) we see that if we have that  $A \lesssim S$  (where  $A$  and  $S$  are positive operators in a type  $II_1$  factor  $\mathcal{M}$  and  $A$  lying in a masa  $\mathcal{A}$ ), then in fact, we have an automorphism  $\phi$  of  $\mathcal{M}$  so that  $E_{\mathcal{A}}(\phi(S)) = A$ . Further, this automorphism is a point  $\|\cdot\|_2$  limit of inner automorphisms. If we additionally assume that  $\mathcal{M}$  is a full factor, for instance one of the free group factors, then we have in fact that there is a unitary  $U$  so that  $E_{\mathcal{A}}(USU^*) = A$ .*

## 6 The Schur-Horn theorem in type $II_\infty$ factors

The Schur-Horn theorem in type  $II_1$  factors allows us to quickly prove an analogous theorem for trace class operators in type  $II_\infty$  factors. One thing to note is that not all masas in type  $II_\infty$  factors admit normal conditional expectations. It is a result of Takesaki[19] that if all masas in a von Neumann algebra admit normal conditional expectations, then the von Neumann algebra is finite. Masas in type  $II_\infty$  factors that do admit normal conditional expectations are generated by their finite projections - We will refer to these as atomic masas in analogy to  $\mathcal{B}(\mathcal{H})$ .

In [4], Arveson and Kadison proved a Schur-Horn theorem for trace class operators in  $\mathcal{B}(\mathcal{H})$ ; We prove an exact analogue of their result here. The proof follows from a routine reduction to the  $II_1$  factor case, which we accomplish by

**Lemma 6.1.** *Let  $\mathcal{A}$  be an atomic masa in a type  $II_\infty$  factor  $\mathcal{M}$  and let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be positive trace class operators so that  $A \prec S$ . Then, there is a unitary and a finite projection  $P$  in  $\mathcal{A}$  so that  $USU^*$  commutes with  $P$  and*

$$AP \prec USU^*P \quad A(I - P) \prec USU^*(I - P)$$

*Proof.* The proof is identical to the first part of the proof of theorem(4.2) and we omit it.  $\square$

The lemma yields a straightforward corollary

**Corollary 6.2.** *Let  $\mathcal{A}$  be an atomic masa in a type  $II_\infty$  factor  $\mathcal{M}$  and let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be positive trace class operators so that  $A \prec S$ . Then, there is a unitary  $U$  and a countable set of orthogonal finite projections  $\{P_n\}$  in  $\mathcal{A}$  so that  $USU^*$  commutes with each projection  $P_n$  and*

$$AP_n \prec USU^*P_n \quad \forall n$$

*Proof.* This is a routine induction argument and we omit it.  $\square$

Recall that for trace class operators in type  $II_\infty$  factors, we have defined  $\mathcal{O}(\mathcal{S})$  as the closure of the unitary orbit in the trace norm, see (6). The Schur Horn theorem for trace class operators in type  $II_\infty$  factors is as follows

**Theorem 6.3.** *Let  $\mathcal{A}$  be an atomic masa in a type  $II_\infty$  factor  $\mathcal{M}$  and let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be positive trace class operators so that  $A \prec S$ . Then, there is an operator  $T \in \mathcal{O}(\mathcal{S})$  so that*

$$E(T) = S$$

where  $E$  is the canonical  $\tau$  preserving conditional expectation onto  $\mathcal{A}$ .

*Proof.* Corollary(6.2) yields us a unitary  $U$  and a countable set of orthogonal projections,  $\{P_n\}$  so that  $USU^* = \sum P_n USU^*$  and so that  $AP_n \prec USU^*P_n$ . Applying theorem(5.6) to each of the  $II_1$  factors  $P_n \mathcal{M} P_n$  yields us a set of operators  $T_n \in \mathcal{O}(P_n USU^*) \in P_n \mathcal{M} P_n$  such that  $E(T_n) = A_n$ . Now, let  $T = \sum_n T_n$  and fix an  $\epsilon > 0$ .

Since  $T_n$  belongs to  $\mathcal{O}(P_n USU^*) \in P_n$ , for each  $n$ , we can find a unitary  $V_n$  in  $P_n \mathcal{M} P_n$  so that

$$\|T_n - V_n(USU^*P_n)V_n^*\|_1 \leq \|T_n - V_n(USU^*P_n)V_n^*\| < \frac{\epsilon}{2^n}.$$

Letting  $V = \sum_n V_n$ , we see that  $\|T - VUSU^*V^*\|_1 < \epsilon$ . Thus,  $T$  belongs to  $\mathcal{O}(\mathcal{S})$  and we are done.  $\square$

Another problem in this context is that of characterizing the images of operators, for instance projections, under the conditional expectation onto an atomic masa. In the case of  $\mathcal{B}(\mathcal{H})$ , there are subtle index type obstructions that pop up[13]. While the diagonals of projections are now completely understood, the situation for operators with spectra containing more than two points is murky, see however[3],[12]. In the type  $II_\infty$  factor case, however, the situation is completely transparent. We first show that any reasonable “diagonal” can be lifted to a projection.

**Theorem 6.4.** *Let  $\mathcal{A}$  be an atomic masa in a type  $II_\infty$  factor  $\mathcal{M}$  and let  $A \in \mathcal{A}$  be a positive contraction. Then, there is a projection  $P$  in  $\mathcal{M}$  so that  $E(P) = A$ .*

*Proof.* Write  $A = \sum_\alpha A Q_\alpha$  where  $Q_\alpha$  are a family of orthogonal finite projections in  $\mathcal{A}$ . Then,  $Q_\alpha \mathcal{M} Q_\alpha$  is a type  $II_1$  factor and we may find a projection  $P_\alpha$  in  $Q_\alpha \mathcal{M} Q_\alpha$  so that  $E(P_\alpha) = A Q_\alpha$ . Then, letting  $P = \sum P_\alpha$ , we have that  $E(P) = A$ .  $\square$

We now turn things around and ask for a characterization of all possible diagonals of a given projection as well as that of positive operators in general. We use the convention that if a positive operator is not trace class, then its trace is  $\infty$ . Argerami and Massey in a recent paper[1] proved approximate theorems in this context, which I am able to improve. First, the result for projections.

**Theorem 6.5.** *Let  $P$  be a projection in a type  $II_\infty$  factor  $\mathcal{M}$  and let  $A \in \mathcal{A}$  be a positive contraction where  $\mathcal{A}$  is an atomic masa. Then, there is a unitary  $U$  such that  $E(UPU^*) = A$  iff  $\tau(P) = \tau(A)$  and  $\tau(I - P) = \tau(I - A)$ .*

*Proof.* If either  $\tau(P)$  or  $\tau(I - P)$  is finite, the theorem follows from theorem(6.3). For the other case, pick an orthogonal family of finite projections  $\{R_\alpha\}$  in  $\mathcal{A}$  summing up to the identity. Decompose  $P = \sum_\alpha P_\alpha$  and  $I - P = \sum Q_\alpha$  so that  $P_\alpha$  and  $Q_\alpha$  are finite projections for every  $\alpha$ , and such that  $\tau(P_\alpha) = \tau(AR_\alpha)$  and  $\tau(P_\alpha) + \tau(Q_\alpha) = \tau(R_\alpha)$ . Pick a unitary  $U$  that conjugates  $P_\alpha + Q_\alpha$  onto  $R_\alpha$  for every  $\alpha$  and by theorem(5.6), pick unitaries  $V_\alpha$  in  $R_\alpha \mathcal{M} R_\alpha$  so that  $E(V_\alpha U(P_\alpha + Q_\alpha) U^* V_\alpha) = AR_\alpha$  for every  $\alpha$ .

Then, if we let  $V = \sum_\alpha V_\alpha$ , we have that

$$E(VUSU^*V^*) = A$$

$\square$

I now extend the above analysis to general positive operators. Let  $S \in \mathcal{M}$  and  $A \in \mathcal{A}$  be positive operators. For there to exist a  $T$  in  $\mathcal{O}(\mathcal{S})$  such that  $E(T) = A$ , it is necessary that  $A \prec S$  (see (2) for the definition of majorization between general positive operators in type  $II_\infty$  factors). However, this is not enough. For example, let  $A$  be a projection such that both  $A$  and  $I - A$  have infinite trace. Let  $\{P_r\}$  be a sequence of trace 1 projections indexed by the rationals in  $\mathbb{Q} \cap (0, 1)$  summing upto  $I$  and let  $S$  be the operator  $S = \sum_{r \in \mathbb{Q} \cap (0, 1)} r P_r$ . Then, for both  $A$  and  $S$ , the upper and lower spectral scales are the constant functions 1 and 0 respectively. It is easy to see that if there is a positive operator  $T$  such that  $E(T) = A$ , then  $T$  must equal  $A$ . However,  $A$  is not in  $\mathcal{O}(\mathcal{S})$ .

Let  $\mathcal{F}(\mathcal{M})$  be the ideal of  $\tau$  finite rank operators,  $\mathcal{F}(\mathcal{M}) = \{x \in \mathcal{M} : \tau(x^*) < \infty\}$  and let  $\mathcal{K}(\mathcal{M}) = \overline{\mathcal{F}(\mathcal{M})}^{||\cdot||}$  be the norm closed two sided ideal of  $\tau$  compact operators[7]. Let  $\mathcal{C}(\mathcal{M})$  be the generalized Calkin algebra  $\mathcal{M}/\mathcal{K}(\mathcal{M})$  and let  $\sigma_e(S)$  and  $\sigma_e(A)$  be the essential spectra of  $S$  and  $A$ , namely the spectra when projected down into  $\mathcal{C}(\mathcal{M})$ . The majorization relation  $A \prec S$  will force  $\sigma_e(A) \subset \text{conv}(\sigma_e(S))$ . The above example shows that we need additional constraints on the essential point spectra of  $A$  and  $S$ . We have the following theorem, whose proof is not too hard - It involves a standard cut and paste argument and a use of theorem(5.6) and we omit it.

**Theorem 6.6.** *Let  $S$  be a positive operator in a type  $II_\infty$  factor  $\mathcal{M}$  and let  $A \in \mathcal{A}$  be a positive operator where  $\mathcal{A}$  is an atomic masa. Then, there is a  $T$  in  $\mathcal{O}(\mathcal{S})$  such that  $E(T) = A$  iff*

1. *We have that  $A \prec S$ . And further,*
2. *If  $\|\sigma_e(A)\| = \|\sigma_e(S)\|$  and if  $\|\sigma_e(A)\|$  belongs to the essential point spectrum of  $A$ , then it belongs to the essential point spectrum of  $S$  as well. And,*
3. *If  $\alpha_e(A) = \alpha_e(S)$  and if  $\alpha_e(A)$  belongs to the essential point spectrum of  $A$ , then it belongs to the essential point spectrum of  $S$  as well.*

## 7 Discussion

It is routine to extend the Schur-Horn theorem to general finite von Neumann algebras. Let  $\mathcal{M}$  be a type  $II_1$  von Neumann algebra and let  $\mathcal{A}$  be a masa in  $\mathcal{M}$ . Instead of working with a tracial state, we must now work with the center valued trace  $\tau$ . Majorization is defined analogously to the case of type  $II_1$  factors. The Schur-Horn theorem in this case is

**Theorem 7.1.** *Let  $\mathcal{A}$  be a masa in a type  $II_1$  von Neumann algebra  $\mathcal{M}$ . If  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  are positive operators with  $A \prec S$ . Then, there is an element  $T \in \mathcal{O}(S)$  such that*

$$E(T) = A.$$

Alternately, we have that

$$E(\mathcal{O}(S)) = \{A \in \mathcal{A} \mid A \prec S\}$$

This can be proved exactly as in the factor case by first getting a local version and then using induction. The proof is a standard application of the direct integral decomposition of  $\mathcal{M}$  into type  $II_1$  factors and an argument analogous to the proof of theorem(5.6) and we omit it.

The situation when it comes to type  $III$  factors is far simpler than that for semifinite factors. The proof is again a simple adaptation of the proof of theorem (5.6) and I omit it.

**Theorem 7.2.** *Let  $\mathcal{A}$  be a masa in a type  $III$  factor  $\mathcal{M}$  that admits a normal conditional expectation. Let  $A \in \mathcal{A}$  and  $S \in \mathcal{M}$  be positive operators. Then the following are equivalent*

1. *There is an operator  $T \in \mathcal{O}(S)$  so that  $E(T) = S$ .*
2. *The following spectral conditions are satisfied*
  - (a)  $\sigma(A) \subset \text{conv}(\sigma(S))$ .
  - (b) *If  $\|S\|$  is in the point spectrum of  $A$ , then it is also in the point spectrum of  $S$ . Similarly for  $\alpha(\sigma(A))$ .*

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